Motion

Jacq Krol

***Abstract*—The most fundamental concept of physics is motion. A good understanding of this concept is essential for any one who tries to understand the laws of physics.**

***Index Terms*—Physics, Force**

I. LAW OF INERTIA

PHYSICS tries to explain the cause of a *change* we observe. The changes that physicists study are changes in *motion*. Motion is observed by the change in *position* of objects relative to an *inertial frame of reference*[1] and with respect to *time*. To study motion a *quantity of motion* should be defined.

It would be desirable that this quantity of motion is not created nor deleted out of nothing, but only transferred from one object to another. The quantity of motion is then conserved in the whole. Only then a change in the motion of an object does not stands on its own, but can always be related to some other change in motion of objects in its environment which is called the cause or action. Physicists try to discover such cause-effect relationships between changes in motion of objects. These relationships should be universal, implying that any *inertial observer*[2] at any place or time, will always observe these same relationships. This universal character of relationships is called the *Principle of Relativity* and the relationships are called *laws*.

Galileo discovered the following law of nature.

Hypothesis 1 (Law of Inertia)**.** *Every object remains being at rest or being in uniform rectilinear motion*[3] *unless it is compelled to change that state by forces impressed on it.*

The motion of an undisturbed object, *inertial motion*, is explained by what we call the *inertia* or *mass* [4] of the object. The inertia is a resistance to change its motion. This law explains inertial motion, so the one thing left to do is to explain *non-inertial motion*.

Isaac Newton published in 1864 his Principia, [1], in which he stated his laws of motion. His first law is just Galilean's law of inertia. He went one step further and formulated a second and third law which explain how an object reacts to an *external force*. Before we formulate Newton's second law we first derive a quantity of motion.

II. MOMENTUM

If two objects with a different mass move with constant velocity, it is observed that the object with the largest mass has the greatest resistance to change its motion. If two objects with the same mass move at different velocities it is also observed that the object with the highest velocity has the greatest resistance to change its motion. The resistance to change motion is therefore proportional to the mass and velocity.

We define the quanity of motion as the product of mass and velocity and call it *momentum*. We use the symbol m for mass, $\boldsymbol{v}$ for velocity and $\boldsymbol{p}$ for momentum. A bold symbol stands for a vector quantity [5] and a normal symbol for a scalar quantity. This gives the following formula for momentum:

$$\boldsymbol{p} = m\boldsymbol{v}$$

III. IMPULSE AND FORCE

In terms of momentum the law of Galileo states that the momentum of a body remains constant unless it is compelled to change by an external influence. If we assume that the mass remains constant then a change in momentum is equivalent to the product of the mass and the change in velocity. In formula form (the symbol Δ stands for a change in a quantity):

$$\Delta\boldsymbol{p} = \Delta(m\boldsymbol{v}) = m\Delta\boldsymbol{v}$$

This $\Delta\boldsymbol{p}$ is the result of the external influence applied to an object. Whenever an influence is applied to an object the impact depends on two factors: the *intensity* and the *duration* of the influence. The duration is measured as Δt and the intensity is a new quantity: **force** with symbol $\boldsymbol{F}$. This gives us the following relation:

$$\boldsymbol{I} = \Delta\boldsymbol{p} = \boldsymbol{F}\Delta t$$

where $\boldsymbol{I}$ is called the **impulse**. From this equation follows the definition of force in terms of quantities we already know:

$$\boldsymbol{F} = \frac{\boldsymbol{I}}{\Delta t} = \frac{\Delta\boldsymbol{p}}{\Delta t}$$

If we substitute the definition of $\Delta\boldsymbol{p}$ into this equation, and assume mass is constant we get the following definition of force:

$$\boldsymbol{F} = m\frac{\boldsymbol{\Delta} v}{\Delta t}$$

This is called the second law of Newton or the law of Action. It defines a force as the change in momentum during some period of time. It also states that a force in physics causes a change of velocity proportional to the force and inversely proportional to the mass:

$$\frac{\boldsymbol{\Delta} v}{\Delta t} = \frac{\boldsymbol{F}}{m}$$

[1] a reference frame with constant velocity

[2] an observer using an inertial reference frame

[3] the same as constant velocity, meaning a constant rate of change of position with respect to time and moving along a straight line

[4] inertia or mass is a quantity that measures the amount of matter of an object

[5] a quantity with a magnitude and a direction in space

IV. DERIVATIVES

Now we first introduce some mathematics to define the concept of change more precisely. The ratio of the change of two quantities is an *average change* of the numerator per unit of the denumerator. For example the ratio of the change in location $\Delta \boldsymbol{x}$ divided by the change in time Δt defines a new quantity, the average change in position per unit of time, which we call the *average velocity*:

$$\bar{\boldsymbol{v}} = \frac{\Delta \boldsymbol{x}}{\Delta t}$$

In the same way we can define a new quantity, the *average acceleration*, which is the ratio of a change in velocity by a change in time:

$$\bar{\boldsymbol{a}} = \frac{\Delta \boldsymbol{v}}{\Delta t}$$

The *average velocity* during Δt does not tell us the speed at any specific instant of time t. The velocity at a specific instant of time, t, is called *instantaneous velocity*. The instantaneous velocity is defined by the ratio of infinitely small changes in position, $d\boldsymbol{x}$,and time,dt,at a specific point in time t.

$$\boldsymbol{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{x}}{\Delta t} = \frac{d\boldsymbol{x}}{dt}$$

In mathematics the instantaneous velocity is called the **first derivative** of $\boldsymbol{x}(t)$ with respect to t and denoted as $\boldsymbol{x}'(t)$, $\dot{\boldsymbol{x}}$, $\frac{d\boldsymbol{x}}{dt}$.

In the same way the *instantaneous acceleration* is defined:

$$\boldsymbol{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{d\boldsymbol{v}}{dt}$$

In mathematics the instantaneous acceleration is called the **second derivative** of $\boldsymbol{x}(t)$ with respect to t and denoted as $\boldsymbol{x}''(t)$, $\ddot{\boldsymbol{x}}$, $\frac{d^2\boldsymbol{x}}{dt^2}$. Note that the second derivative of $\boldsymbol{x}(t)$ is equivalent to the first derivative of $\boldsymbol{v}(t)$.

V. LAW OF ACTION

The definition of force we found before defines an average force. The average force is the product of the mass times the average acceleration:

$$\bar{\boldsymbol{F}} = m\frac{\boldsymbol{\Delta} v}{\Delta t} = m\bar{\boldsymbol{a}}$$

The instantaneous force follows from replacing the average acceleration by the instantaneous acceleration:

$$\boldsymbol{F} = m\boldsymbol{a}$$

This definition is the form in which Newton's second law is mostly expressed in books. Conceptually it is better to remember that force is the first derivative of momentum:

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$$

The reason is that Newton assumed that mass is constant. Relativity theory of Einstein showed that this is not true.

The impact of the force on the momentum of a body is its impulse defined as sum of forces during some amount of time:

$$\boldsymbol{I} = \Delta \boldsymbol{p} = \sum_{t=t_0}^{t_n} \boldsymbol{F}(t)\Delta t = \int \boldsymbol{F} dt$$

VI. WORK AND ENERGY

We have defined impulse as:

$$\boldsymbol{I} = \Delta \boldsymbol{p} = \boldsymbol{F}\Delta t$$

We used Δt to derive Newton's law of Action. Using the quantity $\boldsymbol{F}\Delta t$ enables to calculate a *braking time* of a moving object. Using the quantity $\boldsymbol{F}\Delta \boldsymbol{x}$ enables to calculate a *breaking distance* of a moving object. We define **work**,W:

$$W = \boldsymbol{F}\Delta \boldsymbol{x}$$

Let us analyze this quantity:

$$\boldsymbol{F}\Delta \boldsymbol{x} = \boldsymbol{F}\bar{\boldsymbol{v}}\Delta t = m\Delta \boldsymbol{v}\bar{\boldsymbol{v}}$$

Let us use infinitesimal quantities:

$$\boldsymbol{F} \cdot d\boldsymbol{x} = m\boldsymbol{v} \cdot d\boldsymbol{v}$$

From which we get another definition of force:

$$\boldsymbol{F} = \frac{m\boldsymbol{v}d\boldsymbol{v}}{d\boldsymbol{x}}$$

Force is the derivative of work with respect to $\boldsymbol{x}$:

$$\boldsymbol{F} = \frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{x}}$$

We define **kinetic energy**, T:

$$T = \tfrac{1}{2}m\boldsymbol{v}^2 = \tfrac{1}{2}mv^2$$

and it follows:

$$W = \int dW = \int m\boldsymbol{v}d\boldsymbol{v} = \Delta T = \int \boldsymbol{F} \cdot d\boldsymbol{x}$$

Both momentum and kinetic energy are quantities of motion relating to objects. The rate of change of both quantities is determined by the force applied. So force is the derivative of these quantities with respect to respectively time and position. The total change in momentum is called impulse, and the total change in kinetic energy is called work. Momentum is a vector quantity, and kinetic energy a scalar quantity. If a system is composed of parts the sum of the momenta of the parts equals the momentum of the system as a whole. For kinetic energy this relation does not hold.

VII. CONSERVATION OF MOMENTUM

The Law of Inertia states that an object can't change its own motion. To change the motion of an object A an interaction must occur with another object B. If two objects, A and B, interact the change in momentum observed in A must come from the influence of B, denoted by $\boldsymbol{F}_{B\to A}$, as it can't come from A because of the Law of Inertia and no other object is involved. Along the same line of reasoning the change in momentum of B must come from the influence of A, denoted by $\boldsymbol{F}_{A\to B}$. Based on Newton's second law we have:

$$\boldsymbol{F}_{B\to A} = m_A\boldsymbol{a}_A = \Delta \boldsymbol{p}_A$$
$$\boldsymbol{F}_{A\to B} = m_B\boldsymbol{a}_B = \Delta \boldsymbol{p}_B$$

The momentum of the system of objects A and B is defined as:

$$\boldsymbol{p}_A + \boldsymbol{p}_B$$

If we assume this system is not exposed to an external force, then based on the Law of Inertia, its momentum will not change. But then follows, that for the interaction of A and B within the system:

$$\Delta \boldsymbol{p}_A + \Delta \boldsymbol{p}_B = 0$$

We call this fundamental fact the Law of Conservation of Momentum.

Hypothesis 2 (Conservation of Momentum)**.** *The total momentum of a system of objects before and after any interaction between any two objects is conserved.*

VIII. Conservation of Mechanical Energy

Kinetic energy is not conserved in classical mechanics. Physicists invented a new quantity: **Potential Energy**, V, and defined it as the negative of W (the magnitude of Work),

$$V = -W = -\int F(x)dx$$

The potential energy is used when a force depends on the location in space, $F(x)$. This defines a force field. One the defines total mechanical energy of a system at a point in time, t, as the sum of the kinetic energy and potential energy at t. Kinetic energy depends on the speed, $v(t)$, and potential energy depends on the position, $x(t)$:

$$E(t) = T(v(t)) + V(x(t))$$

Hypothesis 3. *(Conservation of Mechanical Energy) In every motion developing under the action of a force with potential energy V,the sum of the kinetic energy and potential energy is a constant.*

We can show this also mathematically as follows. We must show:

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \frac{\mathrm{d}}{\mathrm{d}t}T(v) + \frac{\mathrm{d}}{\mathrm{d}t}V(x) = 0$$

We have:

$$\frac{\mathrm{d}}{\mathrm{d}t}T(v) = \frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}mv^2) = mva$$
$$\frac{\mathrm{d}}{\mathrm{d}t}V(x) = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\int F(x)dx\right) = -F(x)v$$

Using Newton $F = ma$ we have $\frac{\mathrm{d}}{\mathrm{d}t}T(v) + \frac{\mathrm{d}}{\mathrm{d}t}V(x) = 0$

IX. Law of Action equals Reaction

Based on conservation of momentum follows immediately that for the interaction between two objects A, B:

$$\boldsymbol{F}_{B\to A} = -\boldsymbol{F}_{A\to B}$$

This is called Newton's Third Law or the Law of Action equals Reaction.

X. Summary

The mechanical universe is composed of objects in motion. The quantity of motion is proportional to the mass and velocity of the object. Two quantities of motion, momentum with respect to duration and kinetic energy with respect to distance, are being used. Only an external influence, called a force, applied to an object causes a change in the quantity of motion. A force is the instantaneous change of the quantity of motion, or mathematically the first derivative of the quantity of motion. The total change in the momentum and kinetic energy is then the summation of the force with respect to respectively the duration, (impulse), and distance (work) during which the force is applied. In the classical mechanics of Newton the mass is constant and only the velocity is impacted by the force. Any interaction between two objects, causes a transfer of momentum between the objects of equal magnitude and opposite direction, so that total momentum is always conserved. From this follows the law of action equals reaction in classical mechanics, that a force on an object induces a force of equal magnitude and opposite direction by the object.

References

[1] I. Newton, "The principia: Mathematical principles of natural philosophy," URL, 1687.